Analysis of Three–Parameter Diversely Polarized Array Manifold

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Abstract—An investigative study of three–parameter diversely polarized array manifold is presented in this paper. With a polarization diversity, an extra degree of signal discrimination is provided to an array system allowing the multipath fading to be mitigated and the performance of a communication system to be significantly enhanced. The main objective in this work is to use differential geometry as a tool to analyze the array manifold of this type. A new mathematical framework based on the study of three–parameter array manifold is proposed in conjunction with an analysis of diversely polarized manifold. Various intrinsic parameters such as arclength, area, and volume are defined to provide a fundamental understanding of the geometrical properties and local behavior of the manifold.

I. INTRODUCTION

While array signal processing plays a promising role in the development of future communication systems, it is the concept of array manifold that pinpoints an underlying to this success. Array manifold is the locus of all array response vectors (manifold vectors) over the set of signal parameters. The fact that it is a function of both array geometry and signal constraints makes it an ideal parameter to characterize the whole system.

For many years the principal of the array manifold has been employed in various array processing areas. For instance, it has been used in direction-finding applications to estimate the directions-of-arrival (DOAs) of incoming signals or to design beamformers. The concept is to search over the manifold to find a set of array response vectors that lie in the signal subspace. Examples for these subspace-based algorithms are MUSIC [1] and ESPRIT [2]. Furthermore the intrinsic shape of the manifold plays a major role in the study of array’s ultimate performance in detecting a number of sources and obtaining distinct parameter estimates for each individual source present [3]. The manifold shape also has a direct impact in the presence of manifold ambiguities. This undesirable incident occurs when there exists a linear dependence amongst array response vectors in the signal subspace resulting in spurious extra peaks in the DOA spectrum [3].

Because the fundamental concepts for these applications are directly related to the array manifold properties, a thorough understanding on the geometry of the manifold is proved to be crucial.

So far the geometric studies of array manifold have been confined to only the analysis of the spatial manifolds modeled as a function of a single or two parameters, known as manifold curves and surfaces. In a linear array, the locus of all array response vectors $\mathbf{a}(\theta, \phi)$, which is a function of azimuth $\theta$, represents a curve in complex spaces. Meanwhile, the locus of array response vectors expressed as a function of azimuth and elevation $\mathbf{a}(\theta, \phi)$ in a planar array describes a manifold surface. In [3], the manifold shapes and intrinsic properties of both the curves and surfaces were investigated. However, a study of an array manifold with three parameters, or more, still remains an open research problem, which shall be addressed in this paper.

In the past decade the uses of vector sensors in an array system have gained an increased interest. With polarization diversity an extra degree of signal discrimination is provided. The multipath fading is decreased allowing the reliability and the overall performance of the system to improve. Much of the research in this area focuses on the source localization and polarization estimation using vector sensor array [4][5][6]. However this paper aims to explore the diversely polarized array manifold from a geometric perspective in order to gain an insight into the intrinsic behavior of the manifold.

The organization for the rest of the paper is as follows. In Section II, the differential geometry of manifold curves and surfaces is reviewed. Important terms and parameters to characterize the curves and surfaces are presented. In Section III, a new mathematical framework based on the differential geometry of a three-parameter array manifold is introduced, which is then adopted in Section IV to examine the diversely polarized array manifold. The paper is finally concluded in Section V.

II. MANIFOLD CURVES AND SURFACES

Before proceeding to investigate the spatio-polar manifold let us first review the geometry of spatial manifold curves and surfaces. Our interest lies in the local shape and behavior of the manifolds embedded in complex spaces. For example how a point moves along the manifold curve when there is a small change in the directional parameter, or to observe a local shape of manifold surface (whether it is elliptic, hyperbolic, or a plane.)

Consider an array of $N$ elements, located at $[x_i, y_i, z_i] \in \mathbb{R}^{3N}$ measured in half wavelengths with the array centroid taken as the array reference point $(0, 0, 0)$ receiving co-channel signals from narrow-band point sources. A spatial manifold vector $g(\theta_i, \phi_i)$ that represents the complex array response to a unit amplitude plane wave impinging from the azimuth-elevation $(\theta_i, \phi_i)$ is expressed as

$$g(\theta_i, \phi_i) = \exp(-j [x_i, y_i, z_i] \cdot \mathbf{k}(\theta_i, \phi_i)) \in \mathbb{C}^N$$

(1)

where $\mathbf{k}(\theta_i, \phi_i) = \pi [\cos \theta_i \cos \phi_i, \sin \theta_i \cos \phi_i, \sin \phi_i]^T \in \mathbb{R}^{3 \times 1}$ is the wavenumber vector pointing toward $(\theta_i, \phi_i)$.

For a linear array, the manifold vector given by Equation (1) is simplified to the following single-parameter expression

$$g(\theta) = \exp(-j \pi x \cos \theta)$$

(2)

The locus of all array response vectors $g(\theta), \forall \theta$, in the parameter space $\Omega_\theta$ represents a manifold curve $\mathcal{A}$ embedded in an $N$-dimensional complex spaces formally defined as

$$\mathcal{A} \triangleq \{g(\theta) \in \mathbb{C}^N, \forall \theta : \theta \in \Omega_\theta\}$$

(3)

The most important intrinsic parameter in describing a curve is the arclength $s(\theta)$ that represents the actual physical distance traveled along a curve in $\mathbb{C}^N$, defined as

$$s(\theta) \triangleq \int_0^\theta ||g(\theta)|| d\theta$$

(4)
where a “dot” denotes differentiation with respect to direction $\theta$. The arclength $s$ is an invariant parameter as it can be shown that $\|g'(s)\| = \frac{dg(s)}{ds} = 1$.

Another essential parameter in characterizing the manifold curve is the curvature. It measures how fast the curve gets pulled away from the tangent line. For a linear array the curvature is constant and equal at any point on the curve. This implies that the curve has a hyperbolic shape [3].

Consider now the array manifold of a planar array. The locus of all manifold vectors $a(\theta, \phi)$ over the field-of-view $\Omega$ forms a surface embedded in an $N$-dimensional spaces, known as a manifold surface $M$ and is formally defined as

$$ M = \{ a(\theta, \phi) \in \mathbb{C}^N, \forall (\theta, \phi) : \theta, \phi \in \Omega \} \tag{5} $$

The method required to analyze manifold surfaces is fundamentally different from the previous framework on manifold curves, and is based on the surface differential geometry [3]. One of the most important intrinsic parameter to describe a surface is a matrix known as manifold metric $G$. The metric is a fundamental building block for the derivation of other intrinsic parameters such as arclength and area. It is defined as

$$ G = \begin{pmatrix} \|\dot{a}_\theta\|^2 & \text{Re} \{\dot{a}_\theta^H \dot{a}_\phi\} \\ \text{Re} \{\dot{a}_\theta^H \dot{a}_\phi\} & \|\dot{a}_\phi\|^2 \end{pmatrix} \tag{6} $$

where an element $g_{ij} = \text{Re} \{\dot{a}_i^H \dot{a}_j\}$.

The parameter that provides information about the local shape of the surface at a point $a(\theta, \phi)$ is the Gaussian curvature $K_G$ expressed as

$$ K_G(\theta, \phi) = -\frac{1}{\sqrt{\text{det}(G)}} \left( \frac{d(\text{det}(G))}{d\theta} \left( \Gamma^\phi_{\theta \phi} \right) - \frac{d(\text{det}(G))}{d\phi} \left( \Gamma^\phi_{\theta \phi} \right) \right) \tag{7} $$

where $\Gamma^\phi_{\theta \phi}, \Gamma^\phi_{\theta \phi}$ are elements of the Christoffel symbol metrices of the second kind [7]. The sign of $K_G(\theta, \phi)$ provides an indication of the intrinsic shape of the surface whether it is elliptic if $K_G > 0$, hyperbolic if $K_G < 0$, parabolic or planar if $K_G = 0$.

Furthermore another important concept for the study of manifold surface is the concept of “geodicity”. By definition a geodesic curve is the curve that connects two points on a surface with minimum length. The curve to satisfy this condition must have geodesic curvature (the curve’s first curvature along the tangent plane of the surface) equals to zero. For $\theta$- and $\phi$- curves, the geodesic curvatures are given as

$$ k_{\theta, \phi} \triangleq k_{\theta}(\theta, \phi_0) = \sqrt{\text{det}(G)} \frac{\Gamma^\phi_{\theta \phi}}{g_{\theta \theta}} \tag{8} $$

$$ k_{\phi, \phi} \triangleq k_{\phi}(\theta_0, \phi) = -\sqrt{\text{det}(G)} \frac{\Gamma^\theta_{\phi \phi}}{g_{\phi \phi}} \tag{9} $$

If the two curves are orthogonal and one of them is a geodesic curve, then they constitute a system of geodesic coordinates.

### III. Differential Geometry of Three-Parameter Array Manifold

A new mathematical framework based on the differential geometry of three-parameter array manifold is presented in this section. The array manifold $\mathcal{V}$ is formally defined as

$$ \mathcal{V} = \{ a(x_1, x_2, x_3) \in \mathbb{C}^N, \forall (x_1, x_2, x_3) \in \Omega \} \tag{10} $$

The locus of all array response vectors forms a volume embedded in an $N$- dimensional complex spaces. Our interest lies in the relationship of how a change in volume of parameters $dx_1, dx_2, dx_3$ in $\mathbb{R}^3$ affects the change in shape and properties of the manifold volume $\mathcal{V}$ in $\mathbb{C}^N$.

Note that not every class of manifold can be studied, only certain classes that satisfies the following conditions can be considered [7].

**Definition 1.** A differentiable manifold is a type of topological manifold with a globally defined differential structure. A differentiability class is a classification of function according to the properties of their derivatives. The function is said to be of class $C^k$ if all, up to the $k^{th}$ derivatives exist and are continuous.

An array manifold is said to be smooth if it is the differentiable manifold of class $C^\infty$, i.e. it has derivatives of all orders.

**Definition 2.** A vector function $a = a(x_1, x_2, x_3)$ is a regular parametric representation of the manifold $\mathcal{V}$ if and only if $\forall (x_1, x_2, x_3) \in \Omega$, the derivatives $\dot{a}_x = \frac{da}{dx_1}, \dot{a}_y = \frac{da}{dx_2}, \dot{a}_z = \frac{da}{dx_3}$, and $\dot{a}_x$ exist, are continuous and non-zero with

$$ \text{rank} \{\dot{a}_x, \dot{a}_y, \dot{a}_z\} = 3 \tag{11} $$

This implies the three tangent vectors $\dot{a}_x, \dot{a}_y, \dot{a}_z$ exist everywhere on the manifold, are linearly independent and constitute the covariant basis at the point $a(x_1, x_2, x_3)$. Meanwhile the smooth manifold condition ensures the tangent spaces runs smoothly from point to point throughout the manifold.

It is important to note that our investigative study is limited to an analysis of three-parameter manifold, although the proposed framework can generally be extended to a class of four-parameter array manifold, or more, as long as the prescribed conditions are satisfied. With the number of parameters is restricted to three, the mathematical derivations are simplified and the direct relationship between the manifold $\mathcal{V}$ and the family of manifold surfaces can be drawn.

### A. Manifold Metric

Let us define a Jacobean matrix of tangent vectors,

$$ J = [\dot{a}_x, \dot{a}_y, \dot{a}_z] \in \mathbb{C}^{N \times 3} \tag{12} $$

With a regular parametric representation of $a(x_1, x_2, x_3)$, the manifold metric $G$ is defined as

$$ G = \text{Re} \{J^H J\} \tag{13} $$

$$ = \begin{pmatrix} \|\dot{a}_x\|^2 & \text{Re} \{\dot{a}_x^H \dot{a}_y\} & \text{Re} \{\dot{a}_x^H \dot{a}_z\} \\ \text{Re} \{\dot{a}_y^H \dot{a}_x\} & \|\dot{a}_y\|^2 & \text{Re} \{\dot{a}_y^H \dot{a}_z\} \\ \text{Re} \{\dot{a}_z^H \dot{a}_x\} & \text{Re} \{\dot{a}_z^H \dot{a}_y\} & \|\dot{a}_z\|^2 \end{pmatrix} \tag{14} $$

The notation $g_{ij}$ denotes the $(i^{th}, j^{th})$ element of the metric. This real positive-definite symmetric metric $G$ is a fundamental building block that is essential for a geometric study of three-parameter manifold.

### B. Manifold Volume

Various notations such as angles, lengths of the curves, areas, and volumes can be defined either in terms of the Jacobean matrix $J$, the metric $G$, or its inverse $G^{-1}$. For example an arclength of the curve is given by

$$ s = \int_{t_1}^{t_2} \sqrt{\frac{dx_1^T}{dt} G \frac{dx_1}{dt}} dt \tag{15} $$

where $dx_1 = [dx_1, dx_2, dx_3]^T$. More importantly the manifold volume element is expressed as

$$ v = \iiint_D \sqrt{\text{det}(G)} \, dx_1 dx_2 dx_3 \tag{16} $$
C. Array Manifold as a Family of Surfaces

A three-parameter array manifold can also be treated as a family of surfaces.

**Definition 3.** A constant-parameter surface is defined as the surface that joint all those points on the three-parameter manifold \( \mathcal{V} \) corresponding to a constant value of one of the three parameters \( x_1, x_2, x_3 \).

Three different families of surfaces can be defined depending on which parameter is fixed. For example the \((x_1, x_2)\)-manifold surface is given as

\[
\mathcal{M}_{(x_1, x_2)}|_{x_3 = c} = \left\{ (x_1, x_2, c) \in \mathbb{C}^N : x_1 \in \Omega_{x_1}, \right.
\left. x_2 \in \Omega_{x_2}, x_3 = \text{constant} \right\}
\]  

(17)

Thus the manifold \( \mathcal{V} \) can be observed in three different ways,

\[
\mathcal{V} = \left\{ \mathcal{M}_{(x_1, x_2)}|_{x_3 = c}, \forall c : c \in \Omega_{x_3} \right\}
\]

(18)

\[
= \left\{ \mathcal{M}_{(x_1, x_3)}|_{x_2 = c}, \forall c : c \in \Omega_{x_2} \right\}
\]

(19)

\[
= \left\{ \mathcal{M}_{(x_2, x_3)}|_{x_1 = c}, \forall c : c \in \Omega_{x_1} \right\}
\]

(20)

IV. Three-Parameter Diversely Polarized Manifold

The framework presented in the previous section is now adopted to an analysis of the diversely polarized array manifold. Consider a linear tripole array of \( N \) elements located at \( \Gamma_p = [r_1, \ldots, r_{3N}]^T \in \mathbb{R}^N \) measured in half wavelengths with the array centroid taken as the array reference point \((0, 0, 0)\). In each sensor’s position, there is a set of three mutually perpendicular dipoles receiving transverse electromagnetic (TEM) signals and process them separately. The associated spatio-polar manifold vector is given as \( \hat{A}(\Theta) = A(\theta, \phi) \otimes \hat{q}(\Theta) \in \mathbb{C}^{3N} \)

(21)

where \( \Theta = [\theta, \phi, \gamma, \eta]^T \) denotes the path’s propagation state, and \( \otimes \) is the Kronecker product. The vector \( A(\theta, \phi) \) represents the spatial manifold vector with the azimuth-elevation parameter \((\theta, \phi)\). The polarization parameter \((\gamma, \eta)\) describes the polarization ellipse generated by the electrical field, where \( 0 \leq \gamma \leq \pi/2 \), and \(-\pi \leq \eta \leq \pi \). Furthermore \( \hat{q}(\Theta) = [E_x, E_y, E_z]^T \) is a vector containing the electric field components induced on each dipole by

\[
\hat{q}(\Theta) = \text{diag}(V_x, V_y, V_z)^T \mathcal{T}(\theta, \phi)p(\gamma, \eta)
\]

(22)

where the vector \([V_x, V_y, V_z]^T\) corresponds to the sensor’s sensitivities to unit electric fields in the \( x, y, z \) directions respectively. Meanwhile, \( \mathcal{T}(\theta, \phi) \) is the spherical-to-cartesian transformation matrix given as

\[
\mathcal{T}(\theta, \phi) = \begin{pmatrix}
-\sin \theta & -\cos \theta \sin \phi \\
\cos \theta & -\sin \theta \sin \phi \\
0 & \cos \phi
\end{pmatrix}
\]

(23)

and \( p(\gamma, \eta) = [\cos \gamma, \sin \gamma \ e^{j\eta}]^T \) is the signal’s state of polarization.

For simplicity let us assume that the plane waves are completely polarized and impinging on the array from the same elevation at \( \phi = 0^\circ \). Thus the manifold vector \( \hat{A}(\Theta) \) can be reexpressed as a function of three parameters

\[
\hat{A}(\theta, \gamma, \eta) = M(\theta) \mathcal{T}(\theta)p(\gamma, \eta) \in \mathbb{C}^{3N}
\]

(24)

where \( M(\theta) = [a_x(\theta), a_y(\theta), a_z(\theta)] \in \mathbb{C}^{3N \times 3}, a_x(\theta) = a(\theta) \otimes [V_x, 0, 0]^T, a_y(\theta) = a(\theta) \otimes [0, V_y, 0]^T, a_z(\theta) = a(\theta) \otimes [0, 0, V_z]^T. \) Further assume that the sensor’s sensitivities to electric field are equivalent i.e. \( V_x = V_y = V_z = 1 \).

A. Regular Parametric Representation

The spatio-polar manifold is formally defined as

\[
\mathcal{V} = \left\{ \hat{A}(\theta, \gamma, \eta) \in \mathbb{C}^{3N}, \forall (\theta, \gamma, \eta) \in \Omega \right\}
\]

(25)

It is simple to show that the manifold \( \mathcal{V} \) is a differentiable manifold of class \( C^\infty \) (smooth) and the vector \( \hat{A}(\theta, \gamma, \eta) \) is a regular parametric representation of the manifold.

First, the associated manifold vector \( \hat{A}(\theta, \gamma, \eta) \) has derivatives of all orders and are continuous. Therefore, it is a smooth differentiable manifold of class \( C^\infty \). Second, to prove that the vector \( \hat{A}(\theta, \gamma, \eta) \) is a regular parametric representation of the manifold, it is equivalent to showing that the tangent vectors in the Jacobean matrix \( \mathcal{J} = [\hat{A}_\theta, \hat{A}_\gamma, \hat{A}_\eta] \) exist and its columns are linearly independent, where

\[
\hat{A}_\theta \triangleq \frac{d\hat{A}}{d\theta} = (MT_\theta + M_\theta T)p
\]

(26)

\[
\hat{A}_\gamma \triangleq \frac{d\hat{A}}{d\gamma} = MTP_\gamma
\]

(27)

\[
\hat{A}_\eta \triangleq \frac{d\hat{A}}{d\eta} = MTP_\eta
\]

(28)

Using the proof by contradiction, if \( \hat{A}_\theta, \hat{A}_\gamma, \hat{A}_\eta \) are linearly dependent, then there exists a nonzero vector \( \alpha = [\alpha_1, \alpha_2, \alpha_3]^T \) satisfying the condition,

\[
[(MT_\theta + M_\theta T)p, MTP_\gamma, MTP_\eta] \alpha = 0
\]

(29)

However it can be shown that the expression in Equation (29) cannot be satisfied (except under certain conditions, such as \( \theta = 0^\circ \), or \( 180^\circ \), which are on the boundaries of the manifold).

B. Manifold Metric

With a regular parametric representation of the manifold, the manifold metric \( G \) is defined as

\[
G = \begin{pmatrix}
g_{\theta\theta} & g_{\theta\gamma} & g_{\theta\eta} \\
g_{\theta\gamma} & g_{\gamma\gamma} & g_{\gamma\eta} \\
g_{\theta\eta} & g_{\gamma\eta} & g_{\eta\eta}
\end{pmatrix}
\]

(30)

where the element \( g_{ij} \) is \( \text{Re}\{\hat{A}_i^H \hat{A}_j\} \). To compute the metric, the calculation involves multiplications of \( M, T, p \) and their derivatives which is rather tedious. To ease the computations Table I summarizes some important properties and relationships of \( M, T, p \) and their derivatives. For example, the element \( g_{\theta\eta} \) can be expressed as

\[
g_{\theta\eta} = \text{Re}\{\hat{A}_\theta^H \hat{A}_\eta\} = \text{Re}\{(MT_\theta + M_\theta T)p^H(MTP_\eta)\} = \text{Re}\{(p^H(T_\theta + M_\theta T)p)(MT_\eta)p\} = \text{Re}\{(N_\eta^H + \theta^H T_\eta)MT_\eta\} = 0
\]

(31)

It turns out that all the off-diagonal elements of the metric are zero while the diagonal elements are non-zero. Thus the metric \( G \) is given in the form

\[
G = \begin{pmatrix}
||\hat{A}_\theta||^2, & 0, & 0 \\
0, & ||\hat{A}_\gamma||^2, & 0 \\
0, & 0, & ||\hat{A}_\eta||^2
\end{pmatrix}
\]

(35)

where

\[
g_{\theta\theta} \triangleq ||\hat{A}_\theta||^2 = N \cos^2 \gamma + \pi^2 \sin^2 \theta(V_x^p L_x)
\]

\[
g_{\gamma\gamma} \triangleq ||\hat{A}_\gamma||^2 = N
\]

\[
g_{\eta\eta} \triangleq ||\hat{A}_\eta||^2 = N \sin^2 \gamma
\]
The determinant is found by multiplying all the diagonal elements,
\[ \det G = N^2 \sin^2 \gamma \left[ N \cos^2 \gamma + \pi^2 \sin^2 \theta (x_1^2 + x_2^2) \right] \] (36)

From Equation (16), the term \( \sqrt{\det G} \) was used to detect the change in manifold surface when there is an uncertainty in array position. Similar approach can be performed for the three-parameter case where it can be used as a tool to measure the manifold in terms of the volume variation. This indicates the significance of each sensor that contributes into the array, which is very useful in an array design to search for optimal array position/configuration.

### C. Array Manifold as a Family of \((\gamma, \eta)\)-Manifold Surfaces

Alternatively, the spatio-polar manifold can be described by families of surfaces. The fact that manifold surfaces have been extensively studied in the open literature gives an advantage when considering the manifold as a family of surfaces. Three different families of manifold surfaces can be used to characterize the manifold \( V \) depending on which parameter is fixed. In this work only a family of \((\gamma, \eta)\)-parameter surfaces is considered. This family of surfaces is attractive because the surface is characterized solely by the polarization parameters \((\gamma, \eta)\).

\[ \mathcal{M}_{(\gamma, \eta)|\theta=\theta_0} = \{ A(\theta_0, \gamma, \eta) \in \mathbb{C}^{3N} : 0 \leq \gamma \leq \pi/2, \] 
\[ -\pi \leq \eta < \pi, \theta_0 = \text{constant} \} \] (37)

Meanwhile as the direction \( \theta \) changes, it traces the manifold surface \( \mathcal{M}_{(\gamma, \eta)} \) along the manifold curve in \( \mathbb{C}^{3N} \). The manifold \( V \) considered in this case is defined as

\[ V = \{ \mathcal{M}_{(\gamma, \eta)|\theta=\theta_0}, \forall \theta_0 : \theta_0 \in \Omega_\theta \} \] (38)

Consider when \( \theta = \theta_0 \) the spatio-polar manifold vector is

\[ A(\gamma, \eta) = A(\theta_0, \gamma, \eta) = M(\theta_0)T(\theta_0)p(\gamma, \eta) \] (39)

The matrices \( M(\theta_0) \) and \( T(\theta_0) \) are constant over the parameter \( \theta_0 \). Thus the vector \( A(\gamma, \eta) \) has a direct relationship and can be formed by a linear mapping \( T \) acting on the state of polarization vector \( p(\gamma, \eta) \):

\[ A(\gamma, \eta) = T(p(\gamma, \eta)) = L(\theta_0)p(\gamma, \eta) \] (40)

where \( L(\theta_0) = M(\theta_0)T(\theta_0) \) denotes the linear mapping matrix from \( \mathbb{C}^2 \) to \( \mathbb{C}^{3N} \).

A manifold metric corresponding to the \((\gamma, \eta)\)-parameter surface is given as

\[ G \triangleq \left( \begin{array}{cc} ||A||^2 & \text{Re}(A^H A) \\ \text{Re}(A^H A) & ||A||^2 \end{array} \right) = \left( \begin{array}{cc} N & 0 \\ 0 & N \sin^2 \gamma \end{array} \right) \] (41)

with the determinant \( \det G = N^2 \sin^2 \gamma \). Here the parameter \( \sqrt{\det G} \) does not depend on the array geometry. Thus, uncertainty in the array positions will not have any impact on the changing shape of the manifold surface. The following Theorem presents the geometrical shape of the \((\gamma, \eta)\)-manifold surface.

**Theorem 1.** The \((\gamma, \eta)\)-manifold surface of a tripole array consisting of \( N \) omnidirectional sensors is a spherical with radius \( \sqrt{N} \) embedded in a 3\( N \)-dimensional complex spaces

**Proof:** To show that the surface has a constant and positive Gaussian curvature, let us consider the first and second derivatives of \( A(\gamma, \eta) \) with respect to \( \gamma \) and \( \eta \).

\[ \dot{A}_\gamma = \frac{dA}{d\gamma} = M \dot{p}_\gamma \] (42)

\[ \dot{A}_\eta = \frac{dA}{d\eta} = M \dot{p}_\eta \] (43)

\[ \ddot{A}_\gamma = \frac{ddA}{d\gamma} = \frac{d}{d\gamma} [MP_\eta] = MP_\gamma \] (44)

\[ \ddot{A}_\eta = \frac{ddA}{d\eta} = \frac{d}{d\eta} [MP_\gamma] = MP_\eta \] (45)

\[ \dddot{A}_\gamma = \frac{dddA}{d\gamma} = \frac{d}{d\gamma} [MP_\eta] = MP_\gamma \] (46)

The Christoffel symbol matrices of the first \( C_{1C} \) and second \( C_{2C} \) kind are given by

\[ \Gamma_{1C} = \left( \begin{array}{ccc} \Gamma_{\gamma,\gamma} & \Gamma_{\gamma,\eta} & \Gamma_{\gamma,\eta} \\ \Gamma_{\eta,\gamma} & \Gamma_{\eta,\eta} & \Gamma_{\eta,\eta} \end{array} \right) \] (47)

where \( \Gamma_{i,j,k} \triangleq \text{Re}\{A^H_{ij}A_{jk}\} \) and

\[ \Gamma_{2C} \triangleq -G^{-1}\Gamma_{1C} \left( \begin{array}{ccc} \Gamma_{\gamma,\gamma} & \Gamma_{\gamma,\eta} & \Gamma_{\gamma,\eta} \\ \Gamma_{\eta,\gamma} & \Gamma_{\eta,\eta} & \Gamma_{\eta,\eta} \end{array} \right) \] (48)

It was found that

\[ \Gamma_{1\gamma} = \left( \begin{array}{c} 0 \\ 0 \\ \sin 2\gamma \end{array} \right), \Gamma_{1\eta} = \left( \begin{array}{c} 0 \\ N \sin 2\gamma \\ 0 \end{array} \right) \] (49)

\[ \Gamma_{2\gamma} = \left( \begin{array}{c} 0 \\ 0 \\ \sin \gamma \cos \gamma \end{array} \right), \Gamma_{2\eta} = \left( \begin{array}{c} 0 \\ N \sin \gamma \\ 0 \end{array} \right) \] (50)

From Equation (7), the Gaussian curvature was found to be

\[ K_G(\gamma, \eta) = \frac{1}{\sqrt{\det G}} \left[ d\left( \frac{\sqrt{\det G} \Gamma_{\gamma,\eta}}{d\gamma} \right) - d\left( \frac{\sqrt{\det G} \Gamma_{\eta,\eta}}{d\eta} \right) \right] = \frac{1}{\sqrt{N^2 \sin^2 \gamma}} \left[ \frac{N \sin \gamma \tan \gamma}{d\gamma} - \frac{1}{\tan \gamma} \right] \] (49)

\[ = -\frac{1}{\sqrt{N^2 \sin^2 \gamma}} (\sin \gamma) \] (50)

The curvature is constant and positive everywhere on the surface. This implies that the shape of the \((\gamma, \eta)\)-manifold surface is a spherical with radius \( 1/\sqrt{K_G} = \sqrt{N} \).
Alternatively the Gaussian curvature of the \((\gamma, \eta)\)-manifold surface can be analyzed through the properties formed by the locus of all vectors \(p(\gamma, \eta)\), denoted \(\mathcal{M}_p\). According to Equation (40) the vector \(\mathbf{A}(\gamma, \eta)\) are expressed in a form \(\mathbf{A}(\gamma, \eta) = L(\theta_0)p(\gamma, \eta)\) with \(L^H(\theta_0)L(\theta_0) = N I_2\). This represents a general class of linear mapping. It can be shown that (in a similar approach as in [10]) the Gaussian curvature of the \((\gamma, \eta)\)-surface is equal to \(1/N\) times the curvature of \(\mathcal{M}_p\). The surface \(\mathcal{M}_p\) forms a unit sphere with \(K_G = 1\) in \(\mathbb{C}^2\). Thus the Gaussian curvature for the polarization surface is \(K_G = 1/N\) and this confirms with the earlier result that the surface is of a spherical shape.

**Theorem 2.** The \(\gamma\)- and \(\eta\)- curves on the \((\gamma, \eta)\)-manifold surface are orthogonal and constitute geodesic coordinates.

**Proof:** It is straightforward to see that the two families of curves are orthogonal. This is because \(\text{Re}\{\mathbf{A}_h^H \mathbf{A}_h\} = 0\) as shown in the manifold metric in Equation (41). Furthermore to show that the two families of curves also constitute geodesic coordinates let us find the geodesic curvatures corresponding to both curves. From the expressions of geodesic curvatures in Equations (8)-(9), the curvatures for \(\gamma\) and \(\eta\) curves are given as

\[
k_{\gamma, \gamma} = \pm \frac{\sqrt{\det g_{\gamma}}}{g_{\gamma, \gamma}} \Gamma_{\gamma}^{\gamma} = 0
\]

\[
k_{\gamma, \eta} = -\frac{\sqrt{\det g_{\eta}}}{g_{\eta, \eta}} \Gamma_{\eta}^{\eta} = \frac{1}{N \sqrt{N \tan \gamma}}
\]

Since the \(\gamma\)-curve is the geodesic curve with \(k_{\gamma, \gamma} = 0\) and the curves are orthogonal, they form the geodesic coordinates.

**V. CONCLUSION AND POSSIBLE APPLICATIONS**

An investigative study of three-parameter diversely polarized array manifold was presented in this paper. The aim was to analyze the intrinsic geometry of the manifold in order to gain a more understanding of its local behaviors. In this work a new mathematical framework based on the differential geometry of three-parameter manifold was proposed. This is due to the fact that the spatio-polar manifold is a function of three parameters in which the previous framework cannot be applied.

The locus of all array response vectors over the parameter set defines a manifold volume embedded in complex spaces. Manifold metric is the fundamental building block in describing the manifold. Various intrinsic parameters such as angles, arclength, and volume were defined.

By adopting the proposed framework to the analysis of diversely polarized manifold, several interesting results were obtained. With an assumption that the plane waves are completely polarized and impinged on the array from the same elevation at \(\phi = 0^\circ\), the manifold vector is expressed as a function of three parameters, \(\Theta = [\theta, \gamma, \eta]\). The expression of a volume element was derived in Equation (36), which is a function of both the array geometry and the signal parameters.

On the other hand, the manifold can be visualized as families of surfaces. Consider the manifold as a family of \((\gamma, \eta)\)-manifold surfaces. It was found that the surface has a local spherical shape. Furthermore the \(\gamma\)-curves and \(\eta\)-curves constitute a system of geodesic coordinates.

Due to limited space, the paper can only address the theoretical aspect of the problem where intrinsic geometry of the array manifold were presented. However the results from this study can certainly be adopted to various applications. The parameter \(\sqrt{\det g}\) that represents infinetesimal small volume on the manifold can be used to measure the manifold volume variation when there is uncertainty in the array position. This is a good indication factor for an array design application. Furthermore the result from Theorem 2 shows that the \(\gamma\)-curve is geodesic and \((\gamma, \eta)\) forms the geodesic coordinates. Thus the ultimate detection/resolution thresholds decided based on the separation of two “uncertainty spheres” on a geodesic curve can be derived.

**REFERENCES**